Guided waves:
The two-conductor transmission line

November 20, 2007

Literature


Sophocles Orfanidis, "Electromagnetic Waves and Antennas", electronic book
Outline

How waves can be guided
Lumped Element Circuit Model
Field Analysis
    TEM-Mode
    Coax line
Terminated Lossless Transmission line
    Voltage reflection coefficient
    Return loss
    Standing wave ratio: SWR
    S-parameter
    Input impedance
    Special cases
Lossy Transmission Line
    The low loss line
    Distortion
    Terminated lossy line

Electromagnetic spectrum revisited
Typical waveguiding structures

Waveguides are used to transfer el.mag. power efficiently from one point to another.

The choice of structure is dictated by:
- the desired operating frequency band
- the amount of power to be transferred
- the amount of transmission losses that can be tolerated

Typical waveguiding structures ctd.
- Coaxial cables are used to connect RF-components and typically operate up to 20 GHz.
  However, attenuation is increasing a lot at higher frequencies.
  Another aspect is single-mode operation.
- Two-wire lines are not shielded and radiate already at low frequencies.
- Microstrip lines are used widely in microwave integrated circuits.
- Rectangular waveguides are used up to several hundred 100 GHz.
  At high frequencies difficult to fabricate.
  Useful for high power transmission, e.g. in radar applications.
- Optical fibers operate at optical and infrared wavelength over very wide bandwidths.
  Transmitted power usually at very low levels.
Wave propagation

Consider a system consisting of two conductors extending indefinitely in the $z$-direction.

Associated with the transmission line are:

- $L$ and $C$ describing the lossless operation
- $R$ and $G$ describing the losses

Define

- series impedance $Z$ per unit length
- shunt admittance $Y$ per unit length

\[ Z = R + i\omega L \]
\[ Y = G + i\omega C \]

Key difference between circuit theory and transmission line theory: electrical size

\[ \Delta V = -IZ\Delta z \quad \text{and} \quad \Delta I = -VY\Delta z \]

In the limit $\Delta z \to dz$ we obtain

\[ \frac{dV(z)}{dz} = -ZI(z) \quad \text{and} \quad \frac{dI(z)}{dz} = -YV(z) \]

These are the so called Telegrapher equations

Differentiating and cross substituting, we get wave equations

\[ \frac{d^2 V}{dz^2} = \gamma^2 V \quad \text{and} \quad \frac{d^2 I}{dz^2} = \gamma^2 I \]

where

\[ \gamma^2 = ZY \]

$\gamma$: propagation constant, in general complex

\[ \gamma = \alpha + i\beta \]
\[ \gamma = \pm \sqrt{(R + i\omega L)(G + i\omega C)} \]
\[ \gamma = \pm \sqrt{(RG - LC\omega^2) + i(LG + RC)\omega} \]
Solution

Most general solution of this coupled system:
sum of a forward and backward moving wave
\[ V = v_i e^{i\omega t - \gamma z} + v_r e^{i\omega t + \gamma z} \]
and
\[ I = \frac{\gamma}{Z} (v_i e^{i\omega t - \gamma z} - v_r e^{i\omega t + \gamma z}) \]
where \( v_i e^{i\omega t - \gamma z} \) is the incident
and \( v_r e^{i\omega t + \gamma z} \) is the reflected wave
\[ \rightarrow V = v_i e^{i(\omega t - \beta z)} e^{-\alpha z} + v_r e^{i(\omega t + \beta z)} e^{\alpha z} \]
\( e^{\pm \alpha z} \) describes attenuation
\( \beta = \frac{2\pi}{\lambda} \) is the propagation constant

Characteristic Impedance

Define \( Z_0 = \frac{Z}{\gamma} \): Characteristic impedance

\[ Z_0 = \frac{R + i\omega L}{\sqrt{(RG - L\omega^2) + i(LG + RC)\omega}} = \sqrt{\frac{R + i\omega L}{G + i\omega C}} \]

\( Z_0 \) characterizes the transmission line
\( R, L, G, C \) need to be determined
For the loss less case: \( R = G = 0 \)

\[ \gamma = i\omega \sqrt{LC} \]
\[ \beta = \omega \sqrt{LC} \]
\[ \alpha = 0 \]
\[ Z_0 = \sqrt{\frac{L}{C}} \]
TEM-Mode

Consider transmission line made from two plates

The electric field be only in \( x \)-direction, \( E = E_x \)
The magnetic field be only in \( y \)-direction, \( B = B_y \)
→ TEM-mode, transversal electro magnetic mode
use Faraday’s law

\[
\oint_c \vec{E} \, d\vec{s} = -\frac{\partial}{\partial t} \int_S \vec{B} \, d\vec{a}
\]

and Ampere’s law

\[
\oint_c \vec{B} \, d\vec{s} = \mu_0 \int_S (\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) \, d\vec{a}
\]

TEM-Mode ctd.

\[
\oint_c \vec{E} \, d\vec{s} = \oint_1 \vec{E} \, d\vec{s} + \oint_2 \vec{E} \, d\vec{s} + \oint_3 \vec{E} \, d\vec{s} + \oint_4 \vec{E} \, d\vec{s} = -adz \frac{\partial B_y}{\partial t}
\]

It follows

\[
\frac{\partial V}{\partial z} = -a \frac{\partial B_y}{\partial t} = -\left(\mu_0 \frac{a}{b}\right) \frac{\partial I}{\partial t}
\]

where \( B_y = \mu_0 J_{sz} = \frac{\mu_0 I}{b} \)
TEM-Mode ctd.

\[ bB_y(z) - bB_y(z + dz) = \mu_0 \varepsilon_0 b dz \frac{\partial E_x}{\partial t} \]

and

\[ \frac{\partial I}{\partial z} = -\left( \frac{b}{a} \right) \frac{\partial V}{\partial t} \]

where \( E_x = \frac{V}{a} \) and \( B_y = \mu_0 J_{sz} = \frac{\mu_0}{b} I \)

cross substituting leads to wave equations for \( V \) and \( I \)

\[ \frac{\partial^2 V}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} \quad \frac{\partial^2 I}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 I}{\partial t^2} \]

Coax line

TEM-Mode: E-field is radial and H-field is azimuthal only

\[ E_\rho = \frac{V}{\ln b/a} \frac{1}{\rho} e^{-\gamma z} \quad H_\phi = \frac{V}{\eta \ln b/a} \frac{1}{\rho} e^{-\gamma z} = \frac{I}{2\pi \rho} e^{-\gamma z} \]

In order to find \( Z \) determine \( L, C, R, G \)
Coax line: Characteristic Impedance

\[ L = \frac{\mu}{2\pi} \ln(b/a) \quad C = \frac{2\pi\varepsilon}{\ln(b/a)} \rightarrow Z = \frac{\eta}{2\pi} \ln(b/a) \]

<table>
<thead>
<tr>
<th>type</th>
<th>AWG</th>
<th>( a )</th>
<th>( Z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>RG-6/U</td>
<td>18</td>
<td>0.512</td>
<td>75</td>
</tr>
<tr>
<td>RG-8/U</td>
<td>11</td>
<td>1.150</td>
<td>50</td>
</tr>
<tr>
<td>RG-11/U</td>
<td>14</td>
<td>0.815</td>
<td>75</td>
</tr>
<tr>
<td>RG-58/U</td>
<td>20</td>
<td>0.406</td>
<td>50</td>
</tr>
<tr>
<td>RG-59/U</td>
<td>22</td>
<td>0.322</td>
<td>75</td>
</tr>
<tr>
<td>RG-174/U</td>
<td>26</td>
<td>0.203</td>
<td>50</td>
</tr>
<tr>
<td>RG-213/U</td>
<td>13</td>
<td>0.915</td>
<td>50</td>
</tr>
</tbody>
</table>

- Most commonly used cable is RG-58U with 50Ω
- Home TV cable is usually RG-59U with 75Ω
- Dipole antennas have input impedance of 73Ω → best fed with 75Ω cable
- 50Ω is a compromise between minimum attenuation and maximum power capacity

Coax line: Losses

Loss: Conduction loss, \( \alpha_c \), plus dielectric loss, \( \alpha_d \)

\[ \alpha = \alpha_c + \alpha_d = \frac{1}{2} \left[ \frac{R_s}{\eta} \frac{1}{a} + \frac{1}{b} \ln \frac{b}{a} + \omega \varepsilon'' \eta \right] \]

\[ \eta = \sqrt{\frac{\mu}{\varepsilon'}} \text{ intrinsic impedance of dielectricum.} \quad \eta_0 = 377\Omega \]

\[ R_s = \sqrt{\frac{\omega \mu}{2\sigma}} = \frac{1}{\sigma \delta_s} \quad R_s: \text{ surface resistivity,} \quad \delta_s: \text{ skin depth} \]

- \( \alpha_c \) grows in frequency like \( \sqrt{f} \)
- \( \alpha_d \) grows in frequency like \( f \)
- The smaller \( a \) and \( b \), the higher the losses
- Skin depth \( \delta_s \) at 10 GHz is typical 0.8 \( \mu m \)
- Typical loss tangens, \( \tan \delta = \varepsilon''/\varepsilon' \), for polyethylen or teflon is 0.0004 - 0.0009 up to about 3 GHz
- Attenuation in dB/m is \( \alpha_{dB} = 8.686\alpha \)
dB and Np

Often the ratio of power levels is expressed in decibels (dB):

\[ 10 \log \frac{P_1}{P_2} \text{dB} \]

Memorize:

<table>
<thead>
<tr>
<th>Factor</th>
<th>dB-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

Example: \( 80 = 2 \cdot 2 \cdot 2 \cdot 10 \rightarrow 19\text{dB} = 3 + 3 + 3 + 10 \)

\( 0.02 = 2/10/10 \rightarrow -17\text{dB} = 3 - 10 - 10 \)

Absolute powers expressed in dB notation. Reference power level is 1 mW → 1 mW is 0 dBm

Attenuation, \( e^{-\alpha} \), is often expressed in Nepers, Np

\[ 1\text{Np} = 10 \log e^2 = 8.686\text{dB} \]

\[ 8.686 = 20 / \ln 10 \]

Coax line: Losses

![Graph of dB vs. frequency for RG-8/U and RG-213/U cables](image)

- total
- conductor
- dielectric
- nominal
Loss less transmission line with $Z_0$ is terminated by a load with $Z_L$

$$Z_L = \frac{V(0)}{I(0)} = \frac{v_i + v_r}{v_i - v_r} Z_0$$

$$v_r = \frac{Z_L - Z_0}{Z_L + Z_0} v_i$$

Define: Voltage reflection coefficient $\Gamma$ at load

$$\Gamma = \frac{v_r}{v_i} = \frac{Z_L - Z_0}{Z_L + Z_0}$$
**Guided waves:**

The two-conductor transmission line

How waves can be guided

Lumped Element Circuit Model

Field Analysis

TEM-Mode

Coax line

Terminated Lossless Transmission line

Voltage reflection coefficient

Return loss

Standing wave ratio: SWR

S-parameter

Input impedance

Special cases

Lossy Transmission Line

The low loss line

Distortion

Terminated lossy line

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**RL**

In order that $\Gamma = 0$: load impedance $Z_L$ must be equal to the characteristic impedance of the line $Z_0$.

Such a line is called **matched**

$$V(z) = v_i[e^{-i\beta z} + \Gamma e^{i\beta z}]$$

$$l(z) = \frac{V_i}{Z_0}[e^{-i\beta z} - \Gamma e^{i\beta z}]$$

Average power flow along the line at $z$

$$P_{av} = \frac{1}{2} \text{Re}[V(z)l(z)^*] = \frac{1}{2} \frac{|v_i|^2}{Z_0} (1 - |\Gamma|^2)$$

Mismatched load $\rightarrow$ not all power is delivered to the load

This "loss" is called **return loss, RL**, and is defined in dB

$$RL = -20 \log |\Gamma| \text{ dB}$$

Matched load: $\Gamma = 0 \rightarrow RL = \infty dB$

Total reflection: $|\Gamma| = 1 \rightarrow RL = 0 dB$

---

**SWR**

Mismatched load: $\rightarrow$ reflected waves $\rightarrow$ standing waves

Magnitude of voltage on the line is not constant

$$|V(z)| = |v_i||1 + \Gamma e^{2i\beta z}| = |v_i||1 + \Gamma e^{-2i\beta l}|$$

$$= |v_i||1 + |\Gamma||e^{i(\theta - 2\beta l)}|$$

where $l = -z$ and $\theta$ is the phase of $\Gamma$: $\Gamma = |\Gamma|e^{i\theta}$

Voltage magnitude oscillates with position $z$ along the line

Maximum for $e^{i(\theta - 2\beta l)} = 1 \rightarrow V_{\text{max}} = |v_i|(1 + |\Gamma|)$

Minimum for $e^{i(\theta - 2\beta l)} = -1 \rightarrow V_{\text{min}} = |v_i|(1 - |\Gamma|)$

$V_{\text{max}}$ and $V_{\text{min}}$: measure of mismatch

Define: **standing wave ratio, SWR**

$$SWR = \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

Matched load: $SWR = 1$
Coax line: Certificate

S-parameter or Scattering-matrix
At high frequencies: difficult to measure \( V \) and \( I \)
\rightarrow Instead measure SWR, RL, IL, etc
\rightarrow use network analyzer to measure amplitudes and phases

\[
\begin{pmatrix}
  b_1 \\
  b_2
\end{pmatrix} = \begin{pmatrix}
  S_{11} & S_{12} \\
  S_{21} & S_{22}
\end{pmatrix} \begin{pmatrix}
  a_1 \\
  a_2
\end{pmatrix}
\]

\( S_{ii} \): reflection coefficient seen looking into port \( i \), other ports matched
\( S_{ij} \): transmission coefficient from port \( j \) to port \( i \), other ports matched
Connectors

Requirements for coax-connectors: low SWR, high repeatability after connect-disconnect, mechanical strength, high-order-mode free operation...

Most used connectors at microwave frequencies: N and SMA

N-connector:
named after P.Neill
usable up to 18 GHz
BNC (baby N)

SMA-connector:
usable up to 25 GHz
K-connector up to 40 GHz
2.4mm con. up to 50 GHz

\[ Z_{in} \]

\[ \Gamma \text{ can be generalized to any point } l \text{ on the line} \]

\[ \Gamma(l) = \frac{v_r e^{-i\beta l}}{v_i e^{i\beta l}} = \Gamma(0) e^{-2i\beta l} \]

The impedance seen when looking into the line varies with position
At \( l = -z \) from the load the input impedance, \( Z_{in} \) is

\[ Z_{in} = \frac{V(-l)}{I(-l)} = \frac{v_i [e^{i\beta l} + \Gamma e^{-i\beta l}]}{v_i [e^{i\beta l} - \Gamma e^{-i\beta l}]} Z_0 = \frac{1 + \Gamma e^{-2\beta l}}{1 - \Gamma e^{-2\beta l}} Z_0 \]

Use \( \Gamma = \frac{Z_l - Z_0}{Z_l + Z_0} \) and express \( e^{i\beta l} = \cos(\beta l) + i\sin(\beta l) \) to get:

\[ Z_{in} = Z_0 \frac{Z_L + iZ_0 \tan(\beta l)}{Z_0 + iZ_L \tan(\beta l)} \]

This is the transmission line impedance equation
Impedance in other fields

Barrel displacement and tone quality in the clarinet

For a cylindrical tube, the input impedance is given by (Fletcher and Rossing 1991):

$$ Z_{in} = Z_0 \left( \frac{Z_L \cos kL + jZ_0 \sin kL}{jZ_L \sin kL + Z_0 \cos kL} \right) \quad (1) $$

where $Z_0 = \rho c / S$ is the characteristic impedance, $\rho$ is the air density, $c$ is the speed of sound in air, $S$ is the cross section area, $k$ is the wavenumber, $L$ is the length.

Frequent cases

- Short circuit: $Z_L = 0$
  $\rightarrow \Gamma = -1 \quad SWR = \infty \quad Z_{in} = iZ_0 \tan \beta l$

- Open circuit: $Z_L = \infty$
  $\rightarrow \Gamma = 1 \quad SWR = \infty \quad Z_{in} = -iZ_0 \cot \beta l$

- Half-wave line $l = \lambda / 2$
  $\rightarrow Z_{in} = Z_L \rightarrow \lambda / 2$ -line does not alter or transform $Z_L$

- Quarter-wave line $l = \lambda / 4 + n\lambda / 2$
  $\rightarrow Z_{in} = \frac{Z_0^2}{Z_L}$
  $\rightarrow$ A line with $Z_1$ can be matched to a load or another line with $Z_2$ by inserting a $\lambda / 4$-section with an impedance $Z_0 = \sqrt{Z_1 Z_2}$
  $\rightarrow$ Analogy to optics: matching with a $\lambda / 4$-layer with $n = \sqrt{n_1 n_2}$
Low loss line

In practice: loss due to finite conductivity and lossy dielectric

\[
\gamma = \sqrt{(R + i\omega l)(G + i\omega C)}
\]

\[
= \sqrt{(i\omega l)(i\omega C) \left( 1 + \frac{R}{i\omega L} \right) \left( 1 + \frac{G}{i\omega C} \right)}
\]

\[
\approx i\omega \sqrt{L/C} \sqrt{1 - i \left( \frac{R}{\omega L} + \frac{G}{\omega C} \right)}
\]

and therefore

\[
\alpha \approx \frac{1}{2} \left( R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}} \right) = \frac{1}{2} \left( \frac{R}{Z_0} + GZ_0 \right)
\]

\[
Z_0 = \sqrt{\frac{L}{C}}
\]

\[
\beta \approx \omega \sqrt{L/C}
\]

For low loss lines \(\beta\) and \(Z_0\) as loss less line

Distortionless line

For the lossy line \(\beta\) generally is a complicated function of \(\omega\)

\(\rightarrow\) phase velocity \(v_p = \omega / \beta\) will be different for different frequencies

\(\rightarrow\) distortion of the signal: Dispersion

There is a special case:

\[
\frac{R}{L} = \frac{G}{C}
\]

In this case \(\beta = \omega \sqrt{L/C}\) and \(\alpha\) is independent of frequency. The distortionless line is not loss free, but is capable of passing a pulse or modulation envelope without distortion.
**Z\textsubscript{in} for lossy line**

Lossy transmission line of length $l$ terminated with load impedance $Z_L$

\[
\Gamma(l) = \Gamma e^{-2i\beta l} e^{-2\alpha l} = \Gamma e^{-2\gamma l}
\]

\[
Z_{in} = \frac{Z_L Z_0 + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l}
\]
Maxwell's equations:

\[
\nabla \times \mathbf{E} = -j\omega \mu \mathbf{H} - \mathbf{M} \quad \nabla \cdot \mathbf{D} = \rho
\]

\[
\nabla \times \mathbf{H} = j\omega \varepsilon \mathbf{E} + \mathbf{J} \quad \nabla \cdot \mathbf{B} = 0
\]

Surface resistance and skin depth:

\[
R_s = \sqrt{\frac{\omega \mu}{2\sigma}} \quad \delta_s = \sqrt{\frac{2}{\omega \mu \sigma}}
\]

Input impedance of terminated lossless transmission lines:

\[
Z_{in} = Z_0 \frac{Z_L + j Z_0 \tan \beta \ell}{Z_0 + j Z_L \tan \beta \ell} \quad \text{(arbitrary load)}
\]

\[
Z_{in} = j Z_0 \tan \beta \ell \quad \text{(short-circuited line)}
\]

\[
Z_{in} = -j Z_0 \cot \beta \ell \quad \text{(open-circuited line)}
\]

Relations between load impedance and reflection coefficient:

\[
\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \quad Z_L = Z_0 \frac{1 + \Gamma}{1 - \Gamma}
\]

Definitions of return loss, insertion loss and SWR:

\[
RL = -20 \log |\Gamma|, \quad IL = -20 \log |\Gamma|, \quad \text{SWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|}
\]

Conversion between dB and nepers:

1 neper = 8.686 dB

Elements of the ferrite permeability tensor:

\[
\mu = \mu_0 \left(1 + \frac{\omega_0 \omega_m}{\omega_0^2 - \omega^2}\right) \quad \omega_0 = \mu_0 \gamma H_0
\]

\[
\omega_m = \mu_0 \gamma M_s
\]

\[
\kappa = \mu_0 \frac{\omega_0 \omega_m}{\omega_0^2 - \omega^2} \quad \text{(or 2.8 MHz/Oersted)}
\]

Conversion between some values of reflection coefficient, SWR, and return loss:

| | rl | swr | rl | swr | rl | swr | rl | swr | rl | swr | rl | swr | rl | swr | rl | swr | rl | swr |
| 0.024 | 0.032 | 0.048 | 0.050 | 0.056 | 0.10 | 0.178 | 0.200 | 0.316 | 0.33 |
| 1.05 | 1.07 | 1.10 | 1.11 | 1.12 | 1.22 | 1.43 | 1.50 | 1.92 | 2.00 |
| 32.3 | 30.0 | 26.4 | 26.0 | 25.0 | 20.0 | 15.0 | 14.0 | 10.0 | 9.6 |
# The Effect of VSWR on Transmitted Power

<table>
<thead>
<tr>
<th>VSWR (dB)</th>
<th>Return Loss (dB)</th>
<th>Trans. Loss (dB)</th>
<th>Volt. Refl. Coeff. (%)</th>
<th>Power Trans. (%)</th>
<th>Power Refl. (%)</th>
<th>VSWR (dB)</th>
<th>Return Loss (dB)</th>
<th>Trans. Loss (dB)</th>
<th>Volt. Refl. Coeff. (%)</th>
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